MATH5360 Game Theory
Exercise 5

Assignment 5: 1, 2, 3, 4, 5, 9 (Due: 24 April 2020 (Friday))

1. Let $A = \{A_1, A_2, A_3\}$ be the player set and $X_i = \{0, 1\}$, for $i = 1, 2, 3$, be the strategy set for A_i. Suppose the payoffs to the players are given by the following table.

Strategy	Payoff vector
$(0, 0, 0)$	$(-2, 3, 5)$
$(0, 0, 1)$	$(1, -2, 7)$
$(0, 1, 0)$	$(1, 5, 0)$
$(0, 1, 1)$	$(10, -3, -1)$
$(1, 0, 0)$	$(-1, 0, 7)$
$(1, 0, 1)$	$(-4, 4, 6)$
$(1, 1, 0)$	$(12, -4, -2)$
$(1, 1, 1)$	$(-1, 5, 2)$

   (a) Find the characteristic function of the game.

   (b) Show that the core of the game is empty.

2. Consider a three-person game with characteristic function

$$\begin{aligned} \nu(\{1\}) &= 27 \\ \nu(\{2\}) &= 8 \\ \nu(\{3\}) &= 18 \\ \nu(\{1,2\}) &= 36 \\ \nu(\{1,3\}) &= 50 \\ \nu(\{2,3\}) &= 27 \\ \nu(\{1,2,3\}) &= 60 \end{aligned}$$

   Find the core of the game and draw the region representing the core on the $x_1 - x_2$ plane.

3. Let ν be the characteristic function defined by $\nu(\{1\}) = 3, \nu(\{2\}) = 4, \nu(\{3\}) = 6, \nu(\{1,2\}) = 9, \nu(\{1,3\}) = 12, \nu(\{2,3\}) = 15, \nu(\{1,2,3\}) = 20$.

   (a) Let μ be the $(0, 1)$ reduced form of ν. Find $\mu(\{1,2\}), \mu(\{1,3\}), \mu(\{2,3\})$.

   (b) Find the core of ν and draw the region representing the core on the $x_1 - x_2$ plane.

   (c) Find the Shapley values of the players.

4. Three towns A, B, C are considering whether to built a joint water distribution system. The costs of the construction works are listed in the following table

Coalition	Cost(in million dollars)
$\{A\}$	11
$\{B\}$	7
$\{C\}$	8
$\{A, B\}$	15
$\{A, C\}$	14
$\{B, C\}$	13
$\{A, B, C\}$	20

For any coalition $S \subset \{A, B, C\}$, define $\nu(S)$ to be the amount saved if they build the system together. Find the Shapley values of A, B, C and the amount that each of them should pay if they cooperate.

5. Players 1, 2, 3 and 4 have 45, 25, 15, and 15 votes respectively. In order to pass a certain resolution, 51 votes are required. For any coalition S, define $\nu(S) = 1$ if S can pass a certain resolution. Otherwise $\nu(S) = 0$. Find the Shapley values of the players.

6. Players 1, 2, 3 and 4 have 40, 30, 20, and 10 shares if stocks respectively. In order to pass a certain decision, 50 shares are required. For any coalition S, define $\nu(S) = 1$ if S can pass a certain decision. Otherwise $\nu(S) = 0$. Find the Shapley values of the players.

7. Consider the following market game. Each of the 5 players starts with one glove. Two of them have a right-handed glove and three of them have a left-handed glove. At the end of the game, an assembled pair is worth \$1 to whoever holds it. Find the Shapley value of the players.

8. Let $\mathcal{A} = \{1, 2, 3\}$ be the set of players and ν be a game in characteristic form with

$$\begin{aligned} \nu(\{1\}) &= -a \\ \nu(\{2\}) &= -b \\ \nu(\{3\}) &= -c \\ \nu(\{2,3\}) &= a \\ \nu(\{1,3\}) &= b \\ \nu(\{1,2\}) &= c \\ \nu(\{1,2,3\}) &= 1 \end{aligned}$$

where $0 \le a, b, c \le 1$.

(a) Let μ be the $(0, 1)$ reduced form of ν. Find $\mu(\{1, 2\})$, $\mu(\{1, 3\})$, $\mu(\{2, 3\})$ in terms of a, b, c.

(b) Suppose $a + b + c = 2$. Find an imputation $\mathbf{x}$ of ν which lies in the core $C(\nu)$ in terms of a, b, c and prove that $C(\nu) = \{\mathbf{x}\}$.

9. Aaron (A), Benny (B) and Carol (C) each has to buy a book on Game Theory. The list price of the book is \$200. Alan has a discount card which allow him to buy two books for \$360, and three books for \$480. Benny has a coupon which allows him to

have 20% off for the whole bill. The discount card and coupon can be used at the same time. Let $\nu(S)$ be the amount that a coalition $S \subset \{A, B, C\}$ may save by buying the books together comparing with buying them separately.

(a) Find $\nu(\{A, B\})$, $\nu(\{B, C\})$, $\nu(\{A, C\})$ and $\nu(\{A, B, C\})$

(b) Find $\mu(\{A, B\})$ where μ is the $(0, 1)$ reduced form of ν.

(c) Find the core of ν and draw the region representing the core on the $x_1 - x_2$ plane.

10. Let $a > 0$ be a positive real number. Let $f : [0, a] \to \mathbb{R}$ be a differentiable function such that $f(u) \geq 0$ for any $u \in [0, a]$ and $f(a) = 0$. It is given that the set $\mathcal{R} = \{(u, v) \in \mathbb{R}^2 : 0 \leq u \leq a, 0 \leq v \leq f(u)\}$ is convex. Suppose $(\mu, \nu) \in \mathcal{R}$ and $(\alpha, \beta) = A(\mathcal{R}, (\mu, \nu))$, where A is the arbitration function.

    (a) Show that $f'(\alpha) = -\dfrac{\beta - \nu}{\alpha - \mu}$.

    (b) Let $\mathcal{R} = \{(u, v) \in \mathbb{R}^2 : 0 \leq v \leq 14 + 5u - u^2\}$. Find $(\alpha, \beta) = A(\mathcal{R}, (0, 6))$.

11. Let $\mathcal{A} = \{1, 2, \cdots, N\}$. Prove that for any $i \in \mathcal{A}$

$$\sum_{\{i\} \subset S \subset \mathcal{A}} (N - |S|)!(|S| - 1)! = N!$$

12. Consider an airport game which is a cost allocation problem. Let $N = \{1, 2, \cdots, n\}$ be the set of players. For each $i = 1, 2, \cdots, n$, player i requires an airfield that costs c_i to build. To accommodate all the players, the field will be built at a cost of $\max_{1 \leq i \leq n} c_i$. Suppose all the costs are distinct and $c_1 < c_2 < \cdots < c_n$. Take the characteristic function of the game to be

$$\nu(S) = -\max_{i \in S} c_i$$

    For each $k = 1, 2, \cdots, n$, let $R_k = \{k, k + 1, \cdots, n\}$ and define

$$\nu_k(S) = \begin{cases} -(c_k - c_{k-1}) & \text{if } S \cap R_k \neq \emptyset \\ 0 & \text{if } S \cap R_k = \emptyset \end{cases}$$

    (a) Show that $\nu = \sum_{k=1}^{n} \nu_k$

    (b) Show that for each $k = 1, 2, \cdots, n$, if $i \notin R_k$, then player i is a null player of ν_k.

    (c) Show that for each $k = 1, 2, \cdots, n$, if $i, j \in R_k$, then player i and player j are symmetric players of ν_k.

    (d) Find the Shapley value $\phi_k(\nu)$ of player k, $k = 1, 2, \cdots, n$, of the airport game ν.

13. Let $A = \{1, 2, \ldots, n\}$ and $\nu : \mathcal{P}(A) \to \mathbb{R}$ be the characteristic function defined by

$$\nu(S) = |S| \sum_{i \in S} i$$

where $|S|$ denotes the number of elements in S. Let $\phi_k(\nu)$ be the Shapley value of $k \in A$ in the game (A, ν).

   (a) Show that ν is superadditive.

   (b) For each $i = 1, 2, \ldots, n$, let $\nu_i : \mathcal{P}(A) \to \mathbb{R}$ be the characteristic function defined by

$$\nu_i(S) = \begin{cases} 0, & \text{if } i \notin S \\ i|S|, & \text{if } i \in S \end{cases}.$$

   Let $\phi_k(\nu_i)$ be the Shapley value of $k \in A$ in the game (A, ν_i).

      (i) Show that $\phi_k(\nu_k) = \dfrac{k(n+1)}{2}$.

      (ii) Find $\phi_k(\nu_i)$ for $i \neq k$.

   (c) Using the results in (b), or otherwise, find $\phi_k(\nu)$ in terms of k and n.

14. In a game there are three boxes, Bronze Box, Silver Box and Gold Box. Ada puts \$1,001 into the boxes in any way she likes. The money in Bronze Box will be doubled, the money in Silver Box will be tripled and the money in Gold Box will become 4 times the original amount. Then Bella, without knowing how Ada puts the money, chooses one of the boxes and gets the money inside. Ada will get the money inside the other two boxes.

   (a) How should Ada split the money so that the payoff of Bella are the same no matter what strategy Bella uses.

   (b) Find the strategy of Bella in the Nash's equilibrium.

   (c) Find the expected payoffs of Ada and Bella in the Nash's equilibrium.

   (d) Suppose Ada and Bella decided to cooperate. Using Nash's solution to the bargaining problem and the answer in (c) as the status quo point, determine how much Ada and Bella should get from the boxes.

15. In a money sharing game, three players Alex, Beatrice and Christine put money into a Magic Box. Alex may put from \$0 to \$8, Beatrice may put from \$0 to \$20 and Christine may put from \$0 to \$50. After they put the money, the amount in the Magic Box will be doubled. Then the money in the Magic Box will be evenly distributed to the three players.

   (a) Find the amount that Alex, Beatrice and Christine should put in the Nash equilibrium.

   (b) Find the maximum total profit that Alex and Beatrice may guarantee themselves if they choose to cooperate.

   (c) The three players decide to cooperate. Use Shapley value to find a suitable way to split the money in the Magic Box at the end of the game.